How to... **Solve Systems of Linear Equations**

Given: A set of m variables $\{x_1, x_2, \ldots, x_m\}$ and n linear equations (i.e. equations containing only constant terms or linear terms $a\,x_i$ in the variables x_i).

Wanted: A set $\mathcal{L}$ of solution vectors $\mathbf{x} = (x_1, x_2, \ldots, x_m)^\top$.

Example

We consider the following linear equations:

$$\begin{aligned} x_1 + 2x_2 - 3 &= -2x_4 - x_3 \\ -2x_3 + 8 &= 3x_1 + 6x_2 + 7x_4 \\ 2x_1 + x_3 &= 5 - 4x_2 - 5x_4 \\ -4x_2 + 4 &= 2x_1 + 6x_4 \end{aligned}$$

1 Create matrix representation

Move all terms with variables to one side of the equations and all constant terms to the other such that every equation has the form

$$a_{i,1}\,x_1 + a_{i,2}\,x_2 + \ldots + a_{i,m}\,x_{i,m} = b_i.$$

Create the coefficient matrix $\mathbf{A}$ with entries $a_{i,j}$ and a right hand side vector $\mathbf{b} = (b_1, \ldots, b_n)^\top$. The system of linear equations can now be represented as

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}.$$

Example

We move all variables to the left side and all constants to the right side and obtain

$$\begin{aligned} x_1 + 2x_2 + x_3 + 2x_4 &= 3 \\ -3x_1 - 6x_2 - 2x_3 - 7x_4 &= -8 \\ 2x_1 + 4x_2 + x_3 + 5x_4 &= 5 \\ -2x_1 - 4x_2 \phantom{{}-2x_3} - 6x_4 &= -4 \end{aligned}$$

Thus we get the matrix $\mathbf{A}$ and vector $\mathbf{b}$.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 2 \\ -3 & -6 & -2 & -7 \\ 2 & 4 & 1 & 5 \\ -2 & -4 & 0 & -6 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 3 \\ -8 \\ 5 \\ -4 \end{pmatrix}$$

2 Execute the Gauss Algorithm

a Create augmented matrix

Create the matrix $\left[\mathbf{A} \,\middle|\, \mathbf{b}\right]$ by adding the vector $\mathbf{b}$ to the matrix $\mathbf{A}$.

b Execute Gauss-algorithm steps

Iteratively use one of the following steps

(i) Swap positions of two rows
(ii) Multiply one row by a non-zero factor
(iii) Add (a multiple of) one row to another row

on the augmented matrix $\left[\mathbf{A} \,\middle|\, \mathbf{b}\right]$.

c Stopping criterion

Stop, if the augmented matrix has the following reduced row echelon form:

$$
n \text{ rows} \left\{ \left(\begin{array}{ccccccc|c}
\boxed{1} & * & 0 & * & \cdots & 0 & * & b_1 \\
0 & 0 & \boxed{1} & * & \cdots & 0 & * & b_2 \\
0 & 0 & 0 & 0 & \cdots & 0 & * & b_3 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & * & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & * & b_{r-1} \\
0 & 0 & 0 & 0 & \cdots & \boxed{1} & * & b_r \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{r+1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & 0 & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_n
\end{array} \right) \right.
$$

m columns

The $*$ stands for arbitrary entries. The marked $\boxed{1}$ are the *pivot elements*, the variables correpsonding to these columns are called *pivot variables* or *basis variables*. All other variables (i.e. the variables corresponding to the $*$-columns) are called *non-basis variables*.

We create the augmented matrix

First Pivot Element

$$
\left[\mathbf{A} \,\middle|\, \mathbf{b}\right] = \left(\begin{array}{cccc|c}
\textcircled{1} & 2 & 1 & 2 & 3 \\
-3 & -6 & -2 & -7 & -8 \\
2 & 4 & 1 & 5 & 5 \\
-2 & -4 & 0 & -6 & -4
\end{array} \right).
$$

need zeros here

The top-left element of the matrix is non-zero, so we can use it as the first pivot element. (Otherwise, we could have swapped rows or, in case of a zero-column, moved on to the next column.) We do not need to multiply the first row by some factor since the pivot element is already a 1. In order to obtain zeros below this element, we combine rows in the following way: $\mathrm{II} + 3 \cdot \mathrm{I}$, $\mathrm{III} - 2 \cdot \mathrm{I}$, and $\mathrm{IV} + \mathrm{I}$.

no potential pivot element here

Second Pivot Element

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right).$$

need zeros here

The second column contains only zeros starting from the second column, so there is no candidate for a pivot element in the second column. The next pivot element is the 1 in the third column. (Again, we do not need a multiplication of this row.) We combine the rows in the following way: III + II and IV − 2II and obtain

need zeros here

Second Pivot Element

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Since the pivot element has to be the only non-zero element in its row, we substract the second from the first row (I − II) and get the final tableau

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

There are no more candidates for pivot elements (all further rows are zero rows). The augmented matrix is in the reduced row echelon form. We have 2 pivot elements, and thus 2 basis variables (x_1 and x_3 as the pivots are in the first and third column). The variables x_2 and x_4 are non-basis variables.
The rank of the matrix is 2 (since we have 2 pivot elements).

3 Create the solution set

a Empty solution set

If there is a non-zero b_i value on the right hand side of a otherwise all-zero row, then the solution set is empty, i.e. $\mathcal{L} = \emptyset$.

b Non-empty solution set

If the number of pivot elements (number of basis variables) is r and the number of non-basis variables is $k = m - r$, then create the solution set

$$\mathcal{L} = \left\{ \boldsymbol{x} = \boldsymbol{x}^{(0)} + \lambda_1 \boldsymbol{v}^{(1)} + \ldots + \lambda^{(k)} \boldsymbol{v}^{(k)} \;\middle|\; \lambda_1, \ldots, \lambda_k \in \mathbb{R} \right\}$$

where

(i) $\boldsymbol{x}^{(0)}$ is the vector with (a) the right-hand side values at the position of basis variables and (b) zeros otherwise.

(ii) $\boldsymbol{v}^{(i)}$ is the vector with (a) the *negative* values of the column corresponding to the i-th non-basis variable at the positions of the basis variables, (b) a 1 at the position of the i-th non-basis variables, and (c) zeros otherwise.

The result of the Gauss-algorithm was

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

column for basis solution $\boldsymbol{x}^{(0)}$

← row of first basis variable (x_1)

← row of second basis variable (x_3)

← rhs of 0-rows are zero $\Rightarrow$ solutions exist

column for non-basis solution $\boldsymbol{v}^{(1)}$

column for non-basis solution $\boldsymbol{v}^{(2)}$

So we get the basis solution

$$\boldsymbol{x}^{(0)} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

and the non-basis solutions (these are the basis vectors of the nullspace of the matrix $\mathbf{A}$)

$$\boldsymbol{v}^{(1)} = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{v}^{(2)} = \begin{pmatrix} -3 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

← values from the non-basis column

← values from the non-basis column

← 1 for non-basis variable 2 (x_4)

Hence, the solution set is the following set:

$$\mathcal{L} = \left\{ \boldsymbol{x} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \cdot \begin{pmatrix} -3 \\ 0 \\ 1 \\ 1 \end{pmatrix} \;\middle|\; \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$